Name:
PHYS 2212 Section: April 19-April 23
Recitation 12

Example

Consider a charged sphere of radius R_0 connected to a power source that is modulated so that the volume charge density due to the sphere can be modeled in the spherical coordinate system (r, ϕ, θ) as:

$$\rho(r,t) = \frac{\rho_0}{r^2}\cos(kr - wt)\mathrm{rect}\left(\frac{r}{R_0} - \frac{1}{2}\right) \ ,$$

where ρ_0, k, and ω are positive constants and $\mathrm{rect}(x)$ is the rectangle function, commonly used in electrical engineering:

$$\mathrm{rect}(x) \equiv \begin{cases} 1 & , |x| \leq 1/2 \\ 0 & , |x| > 1/2 \end{cases} \ .$$

Compute the magnetic field $\vec{B}(\vec{x}, t)$ at position $\vec{x} = a\hat{r} + b\hat{\phi} + c\hat{\theta}$, where a,b, and c are constants such that $|\vec{x}| > R_0$.

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Problem

In the circuit shown, if C consists of a pair of parallel, disk-shaped plates of area A and is fully charged to Q_0 at time $t = 0$, compute the magnetic field as a function of time at a point in between the plates but slightly off-axis by a distance d.

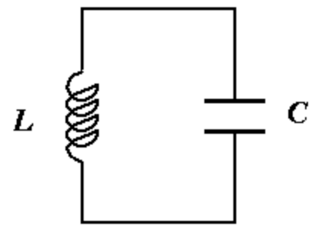

FIG. 1. Circuit for Problem.

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Solution:
Step 1: Physical Intution

The time-dependent charge distribution will give rise to a time-dependent electric field, which will produce a magnetic field. When an electric field induces a magnetic field, the electric and magnetic field lines will be perpendicular to each other. From our experience with Gauss' Law, we can immediately see that the given charge distribution will result in electric field lines that will point in the radial direction. Therefore, the magnetic field lines must point in either the $\hat{\phi}$ or $\hat{\theta}$ directions, or some linear combination of the two. Noticing that the geometry of this problem is spherically symmetric, we can rotate our axes about the origin without altering any of the definitions we've been given. (In addition, the spherical symmetry causes the constants b and c to be irrelevant). As a result, we can arbitrarily set the $\hat{\phi}$ direction to be that of the magnetic field lines; it would have been equally acceptable to use the $\hat{\theta}$ direction as well.

Thus, we are after a magnetic field of the form:

$$\vec{B} = B_\phi\hat{\phi} . \tag{1}$$

Step 2: Gauss' Law

Substituting the charge distribution into the RHS of Gauss' Law for a sphere of radius $r' > R_0$ as we've done in the past gives:

$$\frac{1}{\epsilon_0}\int_V dV\rho = \frac{1}{\epsilon_0}\int_0^{r'} (4\pi r^2 dr)\left(\frac{\rho_0}{r^2}\cos(kr - wt)\text{rect}\left(\frac{r}{R_0} - \frac{1}{2}\right)\right) .$$

In order to proceed, we must determine the edges of the rect function. Setting the absolute value of the argument of the rect function equal to 1/2 results in $0 \leq r \leq R_0$. Consequently:

$$\text{rect}\left(\frac{r}{R_0} - \frac{1}{2}\right) = \begin{cases} 1, 0 \leq r \leq R_0 \\ 0, r < 0 \\ 0, r > R \end{cases} ,$$

so the volume integral becomes:

$$\begin{aligned} \frac{1}{\epsilon_0}\int_0^{r'} (4\pi r^2 dr)\left(\frac{\rho_0}{r^2}\cos(kr - wt)\text{rect}\left(\frac{r}{R_0} - \frac{1}{2}\right)\right) &= \frac{1}{\epsilon_0}\int_0^{R} (4\pi r^2 dr)\left(\frac{\rho_0}{r^2}\cos(kr - wt)\right) , \\ &= \frac{4\pi\rho_0}{\epsilon_0 k}[\sin(kR_0 - \omega t) + \sin(\omega t)] , \end{aligned}$$

where we've applied the chain rule and exploited the odd symmetry of the sine function. The LHS is:

$$\oint_S \vec{E} \cdot \vec{da} = 4\pi r'^2 E_r \ .$$

Equating the sides gives the electric field:

$$\begin{aligned} 4\pi r'^2 E_r &= \frac{4\pi\rho_0}{\epsilon_0 k}[\sin(kR_0 - \omega t) + \sin(\omega t)] \ , \\ E_r(r', t) &= \frac{\rho_0}{\epsilon_0 k r'^2}[\sin(kR_0 - \omega t) + \sin(\omega t)] \ . \end{aligned} \tag{2}$$

Step 3: Ampère-Maxwell law

The Ampère-Maxwell law can be stated as:

$$\oint_C \vec{B} \cdot \vec{ds} = \mu_0 I_{enc} + \epsilon_0\mu_0 \frac{d}{dt}\int_S \vec{E} \cdot \vec{da} \ .$$

Because the point $\vec{x}$ is outside of the sphere, there are no currents involved, allowing us to cross out the first term on the RHS. The simplest Ampèrian loop C we can use that goes through the point $\vec{x}$ is one for which we can deform the bounded surface S to be a hemispherical shell surrounding the charged sphere (Fig. 2). That way, we can compute the flux integral over S in almost the same manner we have been using to compute the closed flux integrals in applying Gauss' Law (recalling also that the radial component of $\vec{x}$ is a):

$$\int_S \vec{E} \cdot \vec{da} = \frac{1}{2}(E_r(r = a)(4\pi a^2) \ ,$$

where the extra factor of one-half comes from the surface being a hemispherical shell rather than a complete shell.

Hence, the RHS of the Ampère-Maxwell law is:

$$\begin{aligned} \epsilon_0\mu_0 \frac{d}{dt}\int_S \vec{E} \cdot \vec{da} &= \epsilon_0\mu_0 \frac{d}{dt}\frac{1}{2}(E_r(a))(4\pi a^2) \ , \\ &= \frac{2\mu_0\pi\rho_0\omega}{k}[\cos(\omega t) - \cos(kR_0 - \omega t)] \ , \end{aligned}$$

where we've substituted in Eq. 2. Now, the RHS:

$$\oint_C \vec{B} \cdot \vec{ds} = (B_\phi(r = a))(2\pi a) \ ,$$

where we've substituted in Eq. 1.

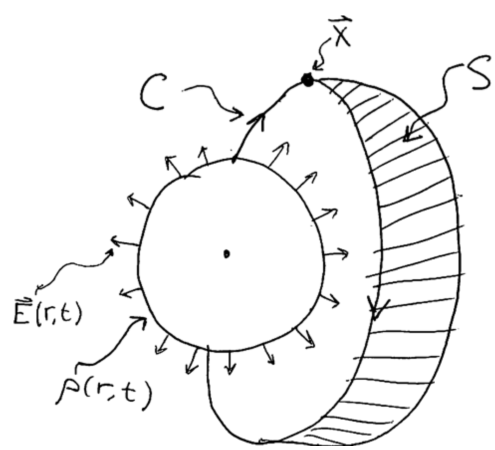

FIG. 2. Application of Ampère-Maxwell Law.

Finally, equating the two sides of the Ampère-Maxwell law gives the final answer:

$$\begin{aligned} (B_\phi(a))(2\pi a) &= \frac{2\mu_0\pi\rho_0\omega}{k}[\cos(\omega t) - \cos(kR_0 - \omega t)] \ , \\ B_\phi(r = a) &= \frac{\mu_0\rho_0\omega}{ka}[\cos(\omega t) - \cos(\omega t - kR_0)] \ , \\ \vec{B}(\vec{x}, t) &= \hat{\phi}\frac{\mu_0\rho_0\omega}{ka}[\cos(\omega t) - \cos(\omega t - kR_0)] \ . \end{aligned}$$

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Problem

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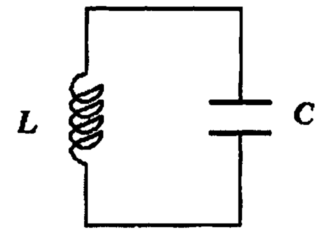
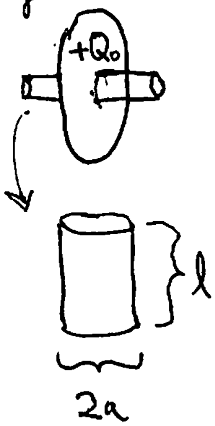


FIG. 1. Circuit for Problem.

$$\frac{Q}{C} - L\frac{dI}{dt} = 0 \text{ (Kirchoff's law)}$$

$$\Rightarrow Q(t) = Q_0 \cos(\omega t) \quad \text{where } \omega = (LC)^{-1/2}.$$

Apply Gauss' Law using pillbox cavity:



$$\oint_S \vec{E}\cdot d\vec{a} = \int_V dV \rho \frac{1}{\varepsilon_0}$$

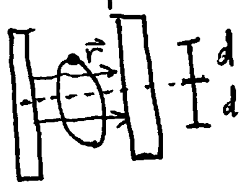
$$E_z \pi a^2 = \int_A dA \frac{Q(t)}{A}\frac{1}{\varepsilon_0}$$

$$= \pi a^2 \frac{Q(t)}{A}\frac{1}{\varepsilon_0}$$

$$= \frac{\pi a^2 Q_0 \cos(\omega t)}{A\varepsilon_0}$$

$$\Rightarrow E_z = \frac{Q_0 \cos(\omega t)}{A\varepsilon_0}$$

Apply Ampère-Maxwell law: $\oint_C \vec{B}\cdot d\vec{s} = \cancel{\mu_0 I_{enc}}^{\,0} + \varepsilon_0\mu_0 \frac{d}{dt}\int_S \vec{E}\cdot d\vec{a}$



$$(B_\phi)\Big|_{r=d}(2\cancel{\pi}\cancel{d}) = \cancel{\varepsilon_0}\mu_0 \frac{d}{dt}\frac{Q_0\cos(\omega t)}{A\cancel{\varepsilon_0}}\cancel{\pi} d^{\cancel{2}}$$

$$B_\phi = \frac{\mu_0 Q_0 \cancel{\pi} d}{A 2\cancel{\pi}}\frac{d}{dt}\cos(\omega t)$$

$$\boxed{B_\phi = \frac{-\mu_0 Q_0 d\,\omega}{2A}\sin(\omega t) \qquad |\vec{B}(r=d)| = \left|+\frac{\mu_0 Q_0 d\,\omega}{2A}\sin(\omega t)\right|}$$

Not enough information provided in problem to specify clockwise or counterclockwise.